Buckling optimization of laminated truncated conical shells subjected to external hydrostatic compression

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Article info

Article history:
Received 15 December 2016
Received in revised form 27 September 2017
Accepted 27 September 2017
Available online 3 October 2017

Keywords:
Optimization
Buckling
Laminated truncated conical shell
Finite element analysis

1. Introduction

The applications of fiber reinforced composite laminated materials in offshore and marine industries have increased rapidly in recent years. These composite structures are commonly subjected to compression in service, which may cause buckling problems [1–10]. The truncated conical shell structures are widely used in offshore platforms, pipelines, submarines and underwater vehicles, which may be subjected to hydrostatic compression. Hence, the buckling of laminated truncated conical shells under hydrostatic compression is of current interest to engineers engaged in offshore and marine engineering practices.

The buckling resistance of laminated truncated conical shells highly depends on end conditions, ply orientations [11–23], and geometric variables such as shell thicknesses, shell lengths, shell radius ratios, cutouts [11,12,14–30] and stiffeners [31–35]. Therefore, for laminated truncated conical shells with a given material system, geometric shape and end condition, the proper selection of appropriate lamination to realize the maximum buckling resistance of the truncated conical shells becomes a crucial problem [36–42]. However, up to present, most optimization works on conical shells have been focused on isotropic materials [43,44] and very few concentrated on laminated materials [41].

Structural optimizations have been popular research areas [45] and lots of them have been focused on laminated materials [46]. There are many optimization methods available today, such as sequential linear programming [37,38,40,42], nonlinear programming [47,48] and reliability-based optimization method [49–52]. Among them, the golden section method [47,48] is simple, efficient and has been successfully applied to many engineering problems. Hence, it is selected in this investigation to perform optimization analyses for the composite truncated conical shells.

In this investigation, optimization of fiber-reinforced laminated truncated conical shells to maximize their critical buckling loads with respect to fiber orientations is performed by using the golden section method. The critical buckling loads of these truncated conical shells with a given material system are maximized with respect to fiber orientations by using the golden section method. Through parametric studies, the influences of the end condition, shell thickness, shell length, shell radius ratio and cutout size on the optimal buckling loads, the associated optimal fiber orientations and the associated buckling modes are demonstrated and discussed.

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shells are modeled by eight-node isoparametric shell elements. For each node, there are three degrees of freedom for displacements and three for rotations. The reduced integration rule together with hourglass stiffness control is employed to formulate the element stiffness matrix [53].

The stress-strain relations for a lamina in the material coordinate (1,2,3) (Fig. 1) can be written as

\[
\begin{bmatrix}
\sigma' \\
\tau' \\
\gamma'
\end{bmatrix} = \begin{bmatrix}
Q_1\\
Q_2\\
Q_5
\end{bmatrix} \begin{bmatrix}
\varepsilon' \\
\gamma'
\end{bmatrix}
\]

where \(\varepsilon' = (\varepsilon_1, \varepsilon_2, \gamma_1, \varepsilon_3, \varepsilon_4)\), \(\gamma' = (\gamma_2, \gamma_3)\), and \(\gamma = (\gamma_{xy}, \gamma_{xz}, \gamma_{yz})\) are the strains in the circumferential, radial, and transverse directions, respectively. The constants equal to zero.

\[
\begin{bmatrix}
\sigma \\
\tau \\
\gamma
\end{bmatrix} = \begin{bmatrix}
\sigma_1 & \sigma_2 & \sigma_3 \\
\tau_1 & \tau_2 & \tau_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{bmatrix} \begin{bmatrix}
\varepsilon \\
\gamma
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix} = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & \sin \theta \cos \theta \\
-2 \sin \theta \cos \theta & \cos^2 \theta & -\sin \theta \cos \theta \\
\cos \theta & \sin \theta & \cos \theta
\end{bmatrix}
\]

where \(\sigma = (\sigma_x, \sigma_y, \tau_{xy})\), \(\tau = (\tau_{xz}, \tau_{yz})\), \(\gamma = (\gamma_{xy}, \gamma_{xz}, \gamma_{yz})\), and \(\theta\) is measured counterclockwise about the \(z\) axis from the element local \(x\)-axis to the material \(1\)-axis. While the element \(x\) axis is in the longitudinal direction of the truncated conical shell, element \(y\) and \(z\) axes are in the circumferential and the radial directions of the truncated conical shell. Let \(\{\sigma\} = (\varepsilon_{xy}, \varepsilon_{xy}, \gamma_{xy})\) be the in-plane strains at the mid-surface of the laminate section, \(\{\kappa\} = (\kappa_x, \kappa_y, \kappa_{xy})\) the curvatures, and \(h\) the total thickness of the section. If there are \(n\) layers in the layup, the stress resultants, \(\{N\} = (N_x, N_y, N_{xy})\), \(\{M\} = (M_x, M_y, M_{xy})\) and \(\{V\} = (V_x, V_y, V_{xy})\), can be defined as

\[
\begin{bmatrix}
\{N\} \\
\{M\} \\
\{V\}
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\{\sigma\} \\
\{\tau\}
\end{bmatrix} dz
\]

\[
= \sum_{j=1}^{n} \int \begin{bmatrix}
\frac{1}{2} (z_j^2 - z_{j-1}^2) [Q_1] & 0 \\
\frac{1}{3} (z_j^3 - z_{j-1}^3) [Q_1] & 0
\end{bmatrix} \begin{bmatrix}
\{\varepsilon_0\} \\
\{\kappa\} \\
\{\gamma\}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\{\sigma_0\} \\
\{\sigma_1\} \\
\{\sigma_2\}
\end{bmatrix}
\]

where \(z_j\) and \(z_{j-1}\) are the distance from the mid-surface of the section to the top and the bottom of the \(j\)-th layer respectively. The \([0]\) is a 3 by 2 matrix with all the coefficients equal to zero.

---

Fig. 1. Material, element and structure coordinates of laminated truncated conical shells.

Fig. 2. The golden section method.
Fig. 3. Truncated conical shells with various end conditions.

(a) Truncated conical shells with various end conditions

(b) Truncated conical shells with central circular cutout

Fig. 4. Effect of end condition and $L/r_1$ ratio on optimal fiber angle and optimal buckling load of $[\pm 90/0/90]_s$ laminated truncated conical shells ($r_1 = 10$ cm, $r_2/r_1 = 0.6$).

(a) optimal fiber angle $\theta_{opt}$ vs. $L/r_1$

(b) optimal buckling load $p_{cr}$ vs. $L/r_1$

Fig. 5. Effect of end condition and $L/r_1$ ratio on optimal fiber angle and optimal buckling load of $[\pm 90/0/0]_s$ laminated truncated conical shells ($r_1 = 10$ cm, $r_2/r_1 = 0.6$).

(a) optimal fiber angle $\theta_{opt}$ vs. $L/r_1$

(b) optimal buckling load $p_{cr}$ vs. $L/r_1$
3. Bifurcation buckling analysis

In the finite-element analysis, a system of nonlinear algebraic equations results in the incremental form:

\[ [K_t]d\{u\} = d\{p\} \]  \hspace{1cm} (7)

where \([K_t]\) is the tangent stiffness matrix, \(d\{u\}\) the incremental nodal displacement vector and \(d\{p\}\) the incremental nodal force vector.

Within the range of elastic behavior, it is known that when the deformation of a structure is small, the nonlinear theory leads to the same critical load as the linear theory [55,56]. Consequently, if only the buckling load is to be determined, the calculation can be greatly simplified by assuming the deformation to be small and the nonlinear terms in the tangent stiffness matrix can be neglected. The linearized formulation then gives rise to a tangent stiffness matrix in the following expression [57]:

\[ [K_t] = [K_L] + [K_s] \]  \hspace{1cm} (8)

where \([K_L]\) is a linear stiffness matrix and \([K_s]\) a stress stiffness matrix. If a stress stiffness matrix \([K_s]_{\text{ref}}\) is generated according to a reference load \(\{p\}_{\text{ref}}\), for another load level \(\{p\}\) with \(\lambda\) a scalar multiplier, it can be written

\[ \{p\} = \lambda\{p\}_{\text{ref}}, \quad [K_s] = \lambda[K_s]_{\text{ref}} \]  \hspace{1cm} (9)

When buckling occurs, the external loads do not change, i.e., \(d\{p\} = 0\). Then the bifurcation solution for the linearized buckling problem may be determined from the following eigenvalue equation:

\[ ([K_t] + \lambda_{cr}[K_s]_{\text{ref}})d\{u\} = \{0\} \]  \hspace{1cm} (10)

where \(\lambda_{cr}\) is an eigenvalue and \(d\{u\}\) becomes the eigenvector defining the buckling mode. The critical load \(p_{cr}\) can be obtained from \(p_{cr} = \lambda_{cr}\{p\}_{\text{ref}}\). In Abaqus, a subspace iteration procedure [58] is used to solve for the eigenvalues and eigenvectors.

4. Golden section method

In this section, the golden section method [47,48] is presented by determining the minimum of the unimodal function \(F\), which is a function of the independent variable \(X\). It is assumed that lower
bound \( X_L \) and upper bound \( X_U \) on \( X \) are known and the minimum can be bracketed (Fig. 2). In addition, it is assumed that the function has been evaluated at both bounds and the corresponding values are \( F_L \) and \( F_U \). Then two intermediate points \( X_1 \) and \( X_2 \) are picked up such that \( X_1 < X_2 \) and the corresponding values of the function at these points are \( F_1 \) and \( F_2 \). Because \( F_1 \) is greater than \( F_2 \), now \( X_1 \) forms a new lower bound and a new set of bounds, \( X_1 \) and \( X_U \) is generated. Now an additional point \( X_3 \) with function value \( F_3 \) is selected. It is clear that \( F_3 \) is greater than \( F_2 \), so \( X_3 \) replaces \( X_U \) as the new upper bound. Repeating this process, the bounds can be narrowed to whatever tolerance is desired.

To determine the method for choosing the interior points \( X_1, X_2, X_3, \ldots \), the values of \( X_1 \) and \( X_2 \) are selected to be symmetric about the center of the interval and satisfying the following expressions:

\[
\frac{X_U - X_2}{X_2} = \frac{X_1 - X_L}{X_1} \tag{11}
\]

\[
\frac{X_1 - X_2}{X_U - X_1} = \frac{X_2 - X_1}{X_2 - X_1} \tag{12}
\]

Let \( \tau \) be a number between 0 and 1. The interior points \( X_1 \) and \( X_2 \) can be defined as

\[
X_1 = (1 - \tau)X_L + \tau X_U \tag{13a}
\]
\[
X_2 = \tau X_L + (1 - \tau)X_U \tag{13b}
\]

Substituting Eqs. (13a) and (13b) into Eq. (12), one obtains

\[
\tau^2 - 3\tau + 1 = 0 \tag{14}
\]

Solving the above equation, one obtains \( \tau = 0.38197 \). The ratio \((1 - \tau)/\tau = 1.61803\) is the famous “golden section” number. For a problem involving the estimation of the maximum of a one-variable function \( F \), the negative of the function can be minimized, that is, minimize \(-F\).

5. Numerical analysis

The shell model in Abaqus program has already been validated by Dassault Systèmes Corporation [53] and by the authors [42] against experimental data of Knight and Starnes [59] and with the numerical result of Stanley [60]. The critical buckling load of the laminated cylindrical panel with a circular cutout obtained from the experimental data of Knight and Starnes [59] is 118.7 kN.
critical buckling loads of the same panel calculated by Abaqus program [53] and Stanley [60] are 112.9 kN (with 4.9% error) and 107.0 kN (with 9.9% error), respectively. The critical buckling mode obtained by Abaqus [53] agrees well with that reported by Stanley [60]. In addition, buckling analyses of grid-stiffened composite cylindrical shells are performed by Wang et al. [32] and Wang et al. [33]. Again, the critical buckling loads obtained by the shell element in Abaqus program agree well with those calculated by the numerical-based smeared stiffener method. Finally, free vibration analyses of grid-stiffened composite cylindrical shells have been performed by Hemmatnezhad et al. [61]. The nature frequencies obtained by the shell element in Abaqus program also agree well with those calculated by the analytical model. As the result, the composite shell model in Abaqus program is proved to be able to predict the critical buckling load of laminated shells with reasonable accuracy.

5.1. Laminated truncated conical shells with various boundary conditions, thicknesses, shell radius ratios and lengths

In this section, laminated truncated conical shells with four types of boundary conditions (Fig. 3) are considered, which are two ends fixed (denoted by FF), left end simply supported and right end fixed (denoted by SF), left end fixed and right end simply supported (denoted by FS), and two ends simply supported (denoted by SS). These conical shells are subjected to external hydrostatic compression \( p \). The ends of these shells are closed and the pressure loads applied at the left end surfaces are transformed into equivalent concentrated ring loads applied to the circular edges along the longitudinal direction of the shell. The radius of the conical shell at the right end, \( r_1 \), is equal to 10 cm and the radius of the shell at the left end, \( r_2 \), varies between 6 cm and 10 cm. The length of the shell \( L \) varies between 10 cm and 40 cm.

The laminate layups of the conical shell are \( \left[ \pm \theta / 90_2 / 0 \right]_{ns} \), where \( n = 2 \) and 6. The thickness of each ply is 0.125 mm. Hence, when \( n = 2 \) the thickness of the shell is 2.5 mm and when \( n = 6 \), the thickness of the shell is 7.5 mm. The lamina consists of Graphite/Epoxy and material constitutive properties are taken from the data of Crawley [62], which are \( E_{11} = 128 \) GPa, \( E_{22} = 11 \) GPa, \( G_{23} = 1.53 \) GPa, \( G_{12} = G_{13} = 4.48 \) GPa, and \( r_{12} = 0.25 \). The convergent analyses of the finite element mesh have been performed by the author [63]. In the analysis, no symmetry simplifi-
<table>
<thead>
<tr>
<th>$L/\eta$</th>
<th>$[\pm \theta/90_2/0]_{2s}$</th>
<th>$[\pm \theta/90_2/0]_{6s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Mode 1" /> $p_{cr} = 7.778 \text{ MPa}$</td>
<td><img src="image2.png" alt="Mode 1" /> $p_{cr} = 89.70 \text{ MPa}$</td>
</tr>
<tr>
<td>2</td>
<td><img src="image3.png" alt="Mode 2" /> $p_{cr} = 4.922 \text{ MPa}$</td>
<td><img src="image4.png" alt="Mode 2" /> $p_{cr} = 59.47 \text{ MPa}$</td>
</tr>
<tr>
<td>3</td>
<td><img src="image5.png" alt="Mode 3" /> $p_{cr} = 3.845 \text{ MPa}$</td>
<td><img src="image6.png" alt="Mode 3" /> $p_{cr} = 46.54 \text{ MPa}$</td>
</tr>
<tr>
<td>4</td>
<td><img src="image7.png" alt="Mode 4" /> $p_{cr} = 2.835 \text{ MPa}$</td>
<td><img src="image8.png" alt="Mode 4" /> $p_{cr} = 39.96 \text{ MPa}$</td>
</tr>
</tbody>
</table>

**Fig. 12.** Optimal buckling modes of $[\pm \theta/90_2/0]_{ms}$ laminated truncated conical shells with two fixed ends and under optimal fiber angles ($r_1 = 10 \text{ cm}$, $r_2 = 6 \text{ cm}$).
<table>
<thead>
<tr>
<th>L/(r_1)</th>
<th>([\pm \theta / 90_2 / 0]_{2s})</th>
<th>([\pm \theta / 90_2 / 0]_{6s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>![Image] (p_{cr} = 6.676) MPa</td>
<td>![Image] (p_{cr} = 78.84) MPa</td>
</tr>
<tr>
<td>2</td>
<td>![Image] (p_{cr} = 3.734) MPa</td>
<td>![Image] (p_{cr} = 46.46) MPa</td>
</tr>
<tr>
<td>3</td>
<td>![Image] (p_{cr} = 2.577) MPa</td>
<td>![Image] (p_{cr} = 31.98) MPa</td>
</tr>
<tr>
<td>4</td>
<td>![Image] (p_{cr} = 2.188) MPa</td>
<td>![Image] (p_{cr} = 27.87) MPa</td>
</tr>
</tbody>
</table>

Fig. 13. Optimal buckling modes of \([\pm \theta / 90_2 / 0]_{6s}\) laminated truncated conical shells with two fixed ends and under optimal fiber angles \((r_1 = 10\ \text{cm}, r_2 = 10\ \text{cm})\).
Citations are made for those laminated truncated conical shells. To find the optimal fiber angle $\theta_{\text{opt}}$ and the associated maximum critical buckling load $P_{\text{cr}}$, the optimization problem can be expressed as:

Maximize : $P_{\text{cr}}(\theta)$ (15a)

Subjected to : $0^\circ \leq \theta \leq 90^\circ$ (15b)

Before the golden section method is carried out, the critical buckling load $P_{\text{cr}}$ of the laminated truncated conical shell is calculated by employing the Abaqus finite element program for every 10° increment in $\theta$ angle to locate the maximum point approximately. Then proper lower and upper bounds are selected so that the critical buckling load is a unimodal function within the search region. Finally, the golden section method is carried out to find the maximum. The optimization process is terminated when an absolute tolerance (the difference of the two intermediate points between the upper bound and the lower bound) $\Delta \theta \leq 0.5$ is reached. After the optimization process is completed, all the maximum points are repeatedly checked by using different sets of initial lower and upper bounds to guarantee that these maximum points are really global maximums.

Figs. 4 and 5 show the optimal fiber angle $\theta_{\text{opt}}$ and the associated optimal critical buckling load $P_{\text{cr}}$ with respect to the $L/r_1$ ratio for the full fiber (FF) shells and the short fiber (SF) shells with various end conditions and with $r_2/r_1 = 0.8$. From Fig. 4a and Fig. 5a, it can be seen that the optimal fiber angle $\theta_{\text{opt}}$ of the laminated truncated conical shell is less sensitive to the boundary conditions. From Fig. 4b and Fig. 5b, one can observe that the optimal critical buckling load $P_{\text{cr}}$ is more sensitive to the end conditions for short conical shells (say $L/r_1 = 1$) and is less sensitive to the end conditions for long conical shells (say $L/r_1 = 4$). This phenomenon is more significant for thick conical shells than for thin conical shells. In addition, the optimal critical buckling load $P_{\text{cr}}$ decreases with the increase of $L/r_1$ ratio. Nevertheless, among these shells under the same geometric configuration, the FF shells have the highest optimal critical buckling loads, then the SF shells and the SS shells. The SS shells have the lowest optimal critical buckling loads. From Fig. 5b, it can be clearly seen that the optimal critical buckling loads of the FF shells and SF shells are very close and the optimal critical buckling loads of the SF shells and SS shells are very close. This confirms that the buckling behavior of the laminated conical shell is governed by the boundary condition at the larger end [16]. Comparing Fig. 5b with Fig. 4b, one can see that the thickness has significantly influence on the optimal critical buckling loads of the laminated conical shells. When the thickness of the shell increases 3 times, the optimal critical buckling loads of...
the shell might increase at least 10 times or more.

Figs. 6 and 7 show the optimal fiber angle \( \theta_{\text{opt}} \) and the associated optimal critical buckling load \( p_{\text{cr}} \) with respect to the \( L/r_1 \) ratio for thin \( [\pm \theta/90_2/0]_{2s} \) and thick \( [\pm \theta/90_2/0]_{6s} \) laminated truncated conical shells with various end conditions and with \( r_2/r_1 = 0.8 \). Figs. 8 and 9 show the optimal fiber angle \( \theta_{\text{opt}} \) and the associated optimal critical buckling load \( p_{\text{cr}} \) with respect to the \( L/r_1 \) ratio for thin \( [\pm \theta/90_2/0]_{2s} \) and thick \( [\pm \theta/90_2/0]_{6s} \) laminated truncated conical shells with various end conditions and with \( r_2/r_1 = 1 \). Generally, these Figs. show similar trends as Figs. 4 and 5.

Figs. 10 and 11 show the optimal fiber angle \( \theta_{\text{opt}} \) and the associated optimal critical buckling load \( p_{\text{cr}} \) with respect to the \( L/r_1 \) ratio for thin \( [\pm \theta/90_2/0]_{2s} \) and thick \( [\pm \theta/90_2/0]_{6s} \) laminated truncated conical shells with two fixed ends and with various \( r_2/r_1 \) ratios. From these Figs., one can notice that the optimal critical buckling loads for both thin and thick conical shells decrease with the increase of \( r_2/r_1 \) ratio. Similar trends are also obtained for laminated truncated conical shells with other boundary conditions [57].

Typical buckling modes for both thin \( [\pm \theta/90_2/0]_{2s} \) and thick \( [\pm \theta/90_2/0]_{6s} \) laminated truncated conical shells with two fixed ends, with \( r_2/r_1 = 0.6 \) and under optimal fiber angle \( \theta_{\text{opt}} \) are given in Fig. 12. It can be observed that the buckling mode of short conical shell might have more wave numbers in the circumferential direction than that of long conical shell. In addition, the buckling mode of thin conical shell might have more wave numbers in the circumferential direction than that of thick conical shell. Fig. 13 shows the buckling modes of thin \( [\pm \theta/90_2/0]_{2s} \) and thick \( [\pm \theta/90_2/0]_{6s} \) laminated truncated conical shells with two fixed ends, with \( r_2/r_1 = 1 \) and under optimal fiber angle \( \theta_{\text{opt}} \). Comparing Fig. 13 with Fig. 12, one could find that the buckling mode of thin conical shell with large \( r_2/r_1 \) ratio would have more wave numbers in the circumferential direction than that with small \( r_2/r_1 \) ratio. Similar trends are also obtained for laminated truncated conical shells with other boundary conditions [63].

5.2. Laminated truncated conical shells \( (L/r_1 = 3) \), with various boundary conditions, thicknesses, shell radius ratios and cutouts

In this section, laminated truncated conical shells with \( L = 30 \) cm and subjected to external hydrostatic compression are analyzed. The radius of the conical shell at the right end, \( r_1 \), is equal to 10 cm and the radius of the shell at the left end, \( r_2 \), varies between 6 cm and 10 cm. The shells contain central circular cutouts with diameter \( d \) varying between 0 cm and 12 cm (Fig. 3b). As before, two laminate layups, \( [\pm \theta/90_2/0]_{2s} \) and \( [\pm \theta/90_2/0]_{6s} \), are selected for analysis. However, the end conditions of the laminated truncated conical shells involve FF and SS only. For the shell
can observe that the optimal critical buckling loads of thin conical shells are greater than those of thick conical shells.

Typical buckling modes for both thin and thick conical shells with cutouts are given in Figs. 19 and 20. These modes show that when the cutout sizes are small, the buckling modes are global (i.e., buckling of entire conical shell). However, when the cutout sizes are large, the buckling modes are more local (i.e., buckling of shell area near hole). Similar trends are also obtained for laminated truncated conical shells with SS boundary conditions.

5.3. Laminated truncated conical shells (d = 8 cm, r2/r1 = 0.6) with various boundary conditions, thicknesses and lengths

In this section, laminated truncated conical shells with r1 = 10 cm, r2 = 6 cm, cutout size d = 8 cm, and subjected to external hydrostatic compression are analyzed. The length of the shell varies between 15 cm and 40 cm. As before, two laminate layups, [±θ/90°2/0]2s and [±θ/90°2/0]6s, and two end conditions, FF and SS, are selected for analysis.

Fig. 21 shows the optimal fiber angle θopt and the associated optimal critical buckling load PCr with respect to the L/r1 ratio for thin [±θ/90°2/0]2s laminated truncated conical shells with cuts, with FF and SS end conditions. From Fig. 21a, it can be seen that the optimal fiber angle θopt of the laminated truncated conical shell is less sensitive to the boundary conditions. When d/r1 < 0.8, the optimal fiber angles of thin conical shells are greater than those of thick conical shells. However, when d/r1 > 0.8, the optimal fiber angles of thin conical shells are less than those of thick conical shells. In addition, the thickness has significantly influenced on the optimal critical buckling loads of the laminated conical shells with cuts. When the thickness of the shell increases 3 times, the optimal critical buckling loads of the shell might increase at least 10 times or more.

Figs. 15 and 16 show the optimal fiber angle θopt and the associated optimal critical buckling load PCr with respect to the d/r1 ratio for thin [±θ/90°2/0]2s laminated truncated conical shells with cuts, with FF and SS end conditions and with r2/r1 = 0.8 and 1. Generally, these Figs. show similar trends as Fig. 14. The exceptions are when r2/r1 = 0.8 and 1, the optimal fiber angles of thin conical shells are always less than that of thick conical shells (Fig. 15b and Fig. 16b).

Figs. 17 and 18 show the optimal fiber angle θopt and the associated optimal critical buckling load PCr with respect to the d/r1 ratio for thin [±θ/90°2/0]2s and thick [±θ/90°2/0]6s laminated truncated conical shells with cuts, with FF and SS end conditions and with various r2/r1 ratios. From Fig. 17a and Fig. 18a, it can be seen that the optimal fiber angles of conical shells with r2/r1 = 0.8 and 1 are very close. From Fig. 17b and Fig. 18b, it can be observed that the optimal critical buckling loads decrease almost linearly with the increase of the r2/r1 ratio.

Typical buckling modes for both thin [±θ/90°2/0]2s and thick [±θ/90°2/0]6s laminated truncated conical shells with cuts, with two fixed ends, with r2/r1 = 0.6 and 1, and under optimal fiber angle θopt are given in Figs. 19 and 20. These modes show that when the cutout sizes are small, the buckling modes are global (i.e., buckling of entire conical shell). However, when the cutout sizes are large, the buckling modes are more local (i.e., buckling of shell area near hole). Similar trends are also obtained for laminated truncated conical shells with SS boundary conditions.

6. Conclusions

Based on the numerical results of this investigation, the following conclusions could be drawn:

1. The optimal fiber angle θopt of the laminated truncated conical shell is less sensitive to the boundary conditions.
Fig. 19. Optimal buckling modes of \( \pm \theta / 90^\circ_2 / 0 \) laminated truncated conical shells with cutouts, with two fixed ends and under optimal fiber angles (\( L = 30 \text{ cm}, r_1 = 10 \text{ cm}, r_2 = 6 \text{ cm} \)).

<table>
<thead>
<tr>
<th>( d / r_1 )</th>
<th>( \pm \theta / 90^\circ_2 / 0 )</th>
<th>( \pm \theta / 90^\circ_2 / 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image1.png" alt="Image" /> ( p_{cr} = 3.845 \text{ MPa} )</td>
<td><img src="image2.png" alt="Image" /> ( p_{cr} = 46.54 \text{ MPa} )</td>
</tr>
<tr>
<td>0.4</td>
<td><img src="image3.png" alt="Image" /> ( p_{cr} = 3.631 \text{ MPa} )</td>
<td><img src="image4.png" alt="Image" /> ( p_{cr} = 44.84 \text{ MPa} )</td>
</tr>
<tr>
<td>0.8</td>
<td><img src="image5.png" alt="Image" /> ( p_{cr} = 3.252 \text{ MPa} )</td>
<td><img src="image6.png" alt="Image" /> ( p_{cr} = 40.49 \text{ MPa} )</td>
</tr>
<tr>
<td>1.2</td>
<td><img src="image7.png" alt="Image" /> ( p_{cr} = 2.793 \text{ MPa} )</td>
<td><img src="image8.png" alt="Image" /> ( p_{cr} = 36.37 \text{ MPa} )</td>
</tr>
<tr>
<td>$d/\eta$</td>
<td>$[\pm\theta / 90_2 / 0]_{2s}$</td>
<td>$[\pm\theta / 90_2 / 0]_{6s}$</td>
</tr>
<tr>
<td>---------</td>
<td>-------------------------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>0</td>
<td><img src="image1.png" alt="Image" /> $P_{cr} = 2.577$ MPa</td>
<td><img src="image2.png" alt="Image" /> $P_{cr} = 31.98$ MPa</td>
</tr>
<tr>
<td>0.4</td>
<td><img src="image3.png" alt="Image" /> $P_{cr} = 2.491$ MPa</td>
<td><img src="image4.png" alt="Image" /> $P_{cr} = 31.09$ MPa</td>
</tr>
<tr>
<td>0.8</td>
<td><img src="image5.png" alt="Image" /> $P_{cr} = 2.340$ MPa</td>
<td><img src="image6.png" alt="Image" /> $P_{cr} = 30.08$ MPa</td>
</tr>
<tr>
<td>1.2</td>
<td><img src="image7.png" alt="Image" /> $P_{cr} = 2.172$ MPa</td>
<td><img src="image8.png" alt="Image" /> $P_{cr} = 28.80$ MPa</td>
</tr>
</tbody>
</table>

**Fig. 20.** Optimal buckling modes of $[\pm\theta / 90_2 / 0]_{in}$ laminated truncated conical shells with cutouts, with two fixed ends and under optimal fiber angles ($L = 30$ cm, $r_1 = 10$ cm, $r_2 = 10$ cm).
2. The optimal critical buckling load $p_{cr}$ is sensitive to the end conditions for short conical shell and is less sensitive to the end conditions for long conical shell. This phenomenon is more significant for thick conical shells than for thin conical shells. The buckling behavior of the laminated truncated conical shell is governed by the boundary condition at the larger end.

3. The optimal critical buckling loads for both thin and thick laminated truncated conical shells with or without cutouts decrease with the increase of $L/r_1$ ratio.

4. The thickness has significantly influence on the optimal critical buckling loads of the laminated truncated conical shells with or without cutouts.

5. The optimal critical buckling loads for both thin and thick laminated truncated conical shells with or without cutouts decrease with the increase of $r_2/r_1$ ratio.

6. The optimal critical buckling loads for both thin and thick laminated truncated conical shells decrease with the increase of the cutout size.

7. The buckling mode of short or thin laminated truncated conical shell might have more wave numbers in the circumferential direction than that of long or thick laminated truncated conical shell.

8. The buckling mode of laminated truncated conical shell with large $r_2/r_1$ ratio would have more wave numbers in the circumferential direction than that with small $r_2/r_1$ ratio.

9. When the cutout sizes are small, the buckling modes of the laminated truncated conical shells are global. When the cutout sizes are large, the buckling modes of the laminated truncated conical shells are more local.

<table>
<thead>
<tr>
<th>$L/ \eta$</th>
<th>$[\pm \theta/90_2/0]_{28}$</th>
<th>$[\pm \theta/90_2/0]_{60}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td><img src="image1.png" alt="Image" /> $p_{cr} = 4.678$ MPa</td>
<td><img src="image2.png" alt="Image" /> $p_{cr} = 58.47$ MPa</td>
</tr>
<tr>
<td>2</td>
<td><img src="image3.png" alt="Image" /> $p_{cr} = 4.046$ MPa</td>
<td><img src="image4.png" alt="Image" /> $p_{cr} = 51.07$ MPa</td>
</tr>
<tr>
<td>3</td>
<td><img src="image5.png" alt="Image" /> $p_{cr} = 3.252$ MPa</td>
<td><img src="image6.png" alt="Image" /> $p_{cr} = 40.49$ MPa</td>
</tr>
<tr>
<td>4</td>
<td><img src="image7.png" alt="Image" /> $p_{cr} = 2.615$ MPa</td>
<td><img src="image8.png" alt="Image" /> $p_{cr} = 36.09$ MPa</td>
</tr>
</tbody>
</table>

Fig. 21. Effect of end condition, thickness and $L/r_1$ ratio on optimal fiber angle and optimal buckling load of $[\pm \theta/90_2/0]_{28}$ laminated truncated conical shells with cutout ($r_1 = 10$ cm, $r_2 = 6$ cm, $d = 8$ cm).

Fig. 22. Optimal buckling modes of $[\pm \theta/90_2/0]_{60}$ laminated truncated conical shells with cutouts, with two fixed ends and under optimal fiber angles ($r_1 = 10$ cm, $r_2 = 6$ cm, $d = 8$ cm).
Acknowledgment

This research work was financially supported by the Ministry of Science and Technology, Republic of China under Grant MOST 103-2221-E-006-275.

References